THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4030 Differential Geometry
21 November, 2024 Tutorial

1. Compute the Gaussian curvature and mean curvature of the surface $z = axy$ where $a \neq 0$.

2. Let S be a surface with isothermal parameterization

$$g = \begin{pmatrix} \exp(2f) & 0 \\ 0 & \exp(2f) \end{pmatrix}$$

where f is a given smooth function. Using the fact that in this parameterization

$$\Gamma_{ij}^k = \delta_{ki} f_j + \delta_{kj} f_i - \delta_{ij} f_k,$$

compute the Gaussian curvature K. Here, δ_{ij} is the Kronecker delta.

3. In the case of a surface of revolution

$$X(u, v) = (f(v)\cos u, f(v)\sin u, g(v)), \quad f(v) > 0$$

the geodesic equations are

$$\begin{aligned} u'' + \frac{2ff_v}{f^2} u'v' &= 0 \\ v'' - \frac{ff_v}{f_v^2 + g_v^2}(u')^2 + \frac{f_v f_{vv} + g_v g_{vv}}{f_v^2 + g_v^2}(v')^2 &= 0. \end{aligned}$$

(a) For a surface of revolution, let α be a geodesic intersecting a parallel $\beta(t) = X(t, c)$ (where c is a constant) at an angle θ. Let r be the distance to the axis of revolution. Prove Clairaut's relation:

$$r\cos\theta = f(v(t))\cos\theta = \text{const}.$$

(b) Using Clairaut's relation, show that on a torus T parameterized by

$$X(u, v) = ((r\cos v + a)\cos u, (r\cos v + a)\sin u, r\sin v), \quad 0 < r < a$$

if a geodesic α is tangent to the parallel $\beta(t) = X(t, \frac{\pi}{2})$ at some point t_0, then α is entirely contained in the region of T given by $-\dfrac{\pi}{2} \leq v \leq \dfrac{\pi}{2}$.

1. Compute the Gaussian curvature and mean curvature of the surface $z = axy$ where $a \neq 0$.

2. Let S be a surface with isothermal parameterization

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

$$g = \begin{pmatrix} \exp(2f) & 0 \\ 0 & \exp(2f) \end{pmatrix}$$

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Derive this from formula for Γ^k_{ij} $\rightarrow$ $\Gamma^k_{ij} = \delta_{ki} f_j + \delta_{kj} f_i - \delta_{ij} f_k$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

compute the Gaussian curvature K. Here, δ_{ij} is the Kronecker delta.

Sol'n: $K = \frac{1}{2} g^{ij} \left(\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^q_{ij} \Gamma^k_{kq} - \Gamma^q_{ik} \Gamma^k_{jq} \right)$.

Note that $g^{-1} = \begin{bmatrix} \exp(-2f) & 0 \\ 0 & \exp(-2f) \end{bmatrix}$.

So $g^{ij} \neq 0$ only when $i=j$

$$K = \frac{1}{2} g^{11} \left(\partial_k \Gamma^k_{11} - \partial_1 \Gamma^k_{1k} + \Gamma^q_{11} \Gamma^k_{kq} - \Gamma^q_{1k} \Gamma^k_{1q} \right) + \frac{1}{2} g^{22} \left(\partial_k \Gamma^k_{22} - \partial_2 \Gamma^k_{2k} + \Gamma^q_{22} \Gamma^k_{kq} - \Gamma^q_{2k} \Gamma^k_{2q} \right).$$

$\Gamma^1_{11} = \delta_{11} f_1 + \delta_{11} f_1 - \delta_{11} f_1 = f_1$

$\Gamma^1_{12} = \delta_{11} f_2 + \cancel{\delta_{12}}^{0} f_1 - \cancel{\delta_{12}}^{0} f_1 = f_2$.

$\Gamma^1_{21} = f_2$.

$\Gamma^1_{22} = \cancel{\delta_{12}} f_2 + \cancel{\delta_{12}} f_2 - \delta_{22} f_1 = -f_1$

$\Gamma^2_{11} = \delta_{21} f_1 + \delta_{21} f_1 - \delta_{11} f_2 = -f_2$

$\Gamma^2_{12} = f_1$, $\Gamma^2_{21} = f_1$, $\Gamma^2_{22} = f_2$.

$$\partial_k \Gamma^k_{11} = \partial_1 \Gamma^1_{11} + \partial_2 \Gamma^2_{11} = f_{11} - f_{22}$$

$$\partial_1 \Gamma^k_{1k} = \partial_1 \Gamma^1_{11} + \partial_1 \Gamma^2_{12} = 2f_{11}$$

$$\begin{aligned}
\Gamma^q_{11} \Gamma^k_{kq} &= \Gamma^1_{11} \Gamma^k_{k1} + \Gamma^2_{11} \Gamma^k_{k2} \\
&= \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^2_{22} \\
&= f_1^2 + f_1^2 - f_2^2 - f_2^2 \\
&= 2f_1^2 - 2f_2^2.
\end{aligned}$$

$$\begin{aligned}
\Gamma^q_{1k} \Gamma^k_{1q} &= \Gamma^1_{1k} \Gamma^k_{11} + \Gamma^2_{1k} \Gamma^k_{12} \\
&= \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^1_{12} + \Gamma^2_{12}\Gamma^2_{12} \\
&= f_1^2 + f_2(-f_2) - f_2 f_2 + f_1^2 \\
&= 2f_1^2 - 2f_2^2.
\end{aligned}$$

So $\frac{1}{2}g^{11}\left(\partial_k \Gamma^k_{11} - \partial_1 \Gamma^k_{1k} + \Gamma^q_{11}\Gamma^k_{kq} - \Gamma^q_{1k}\Gamma^k_{1q}\right)$

$$= \frac{1}{2}\exp(-2f)\left(f_{11} - f_{22} - 2f_{11} + \cancel{2f_1^2} - \cancel{2f_2^2} - \cancel{2f_1^2} + \cancel{2f_2^2}\right)$$

$$= -\frac{1}{2}\exp(-2f)(f_{11} + f_{22}).$$

Similarly, $\frac{1}{2}g^{22}\left(\partial_k \Gamma^k_{22} - \partial_2 \Gamma^k_{2k} + \Gamma^q_{22}\Gamma^k_{kq} - \Gamma^q_{2k}\Gamma^k_{2q}\right).$

$$= -\frac{1}{2}\exp(-2f)(f_{11} + f_{22}).$$ Check this!

$$K = \exp(-2f)\Delta f.$$

3. In the case of a surface of revolution

$$X(u,v) = (f(v)\cos u, f(v)\sin u, g(v)), \quad f(v) > 0$$

the geodesic equations are

derive these from general geodesic equation

$$u'' + \frac{2ff_v}{f^2}u'v' = 0$$
$$v'' - \frac{ff_v}{f_v^2 + g_v^2}(u')^2 + \frac{f_v f_{vv} + g_v g_{vv}}{f_v^2 + g_v^2}(v')^2 = 0.$$

$u_k'' + \Gamma^k_{ij} u_i' u_j' = 0$

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